The Effect of Network Topology on the Spread of Epidemics

Complex Network: Level M/7 Presentation

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Background

How do the epidemics spread through a network?

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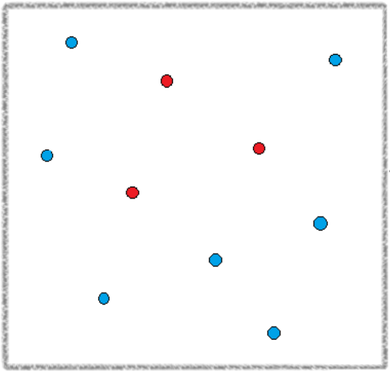

- **blue circle** -- healthy but susceptible
- **red circle** -- infected

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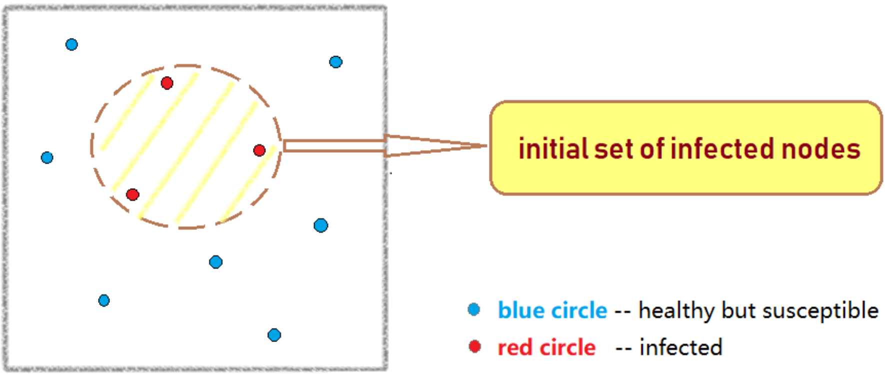

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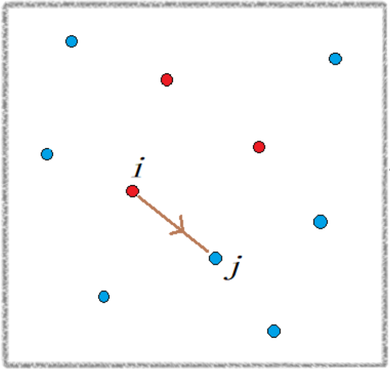

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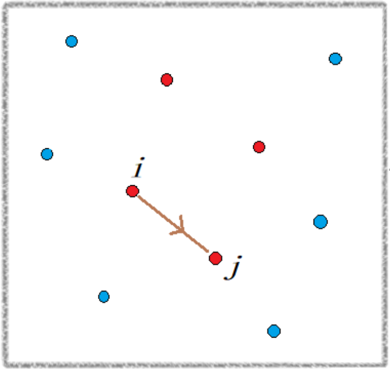

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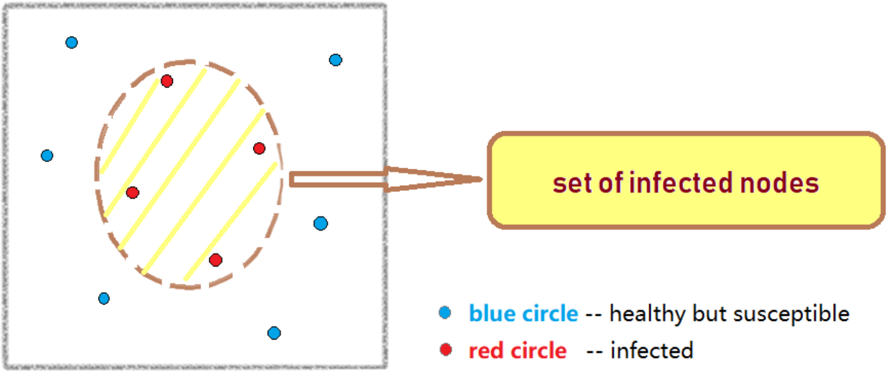

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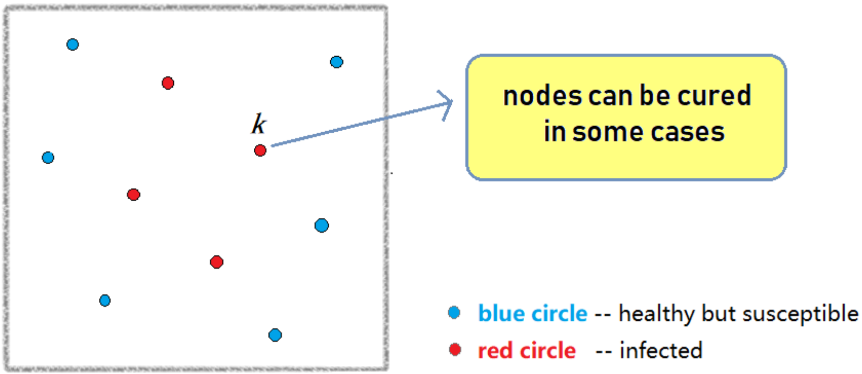

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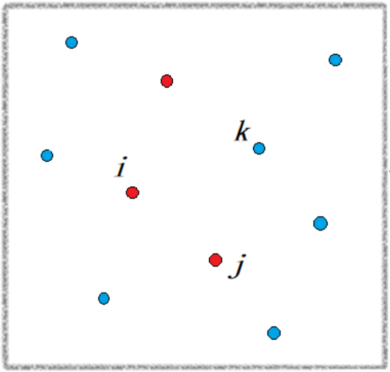

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Aim

Spread of epidemics in a network :

Explore how the **ratio of cure to infection rates** affects the **mean epidemic lifetime**

Motivation

Explore the topological properties in a network :

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- spread of epidemics ;
- spread of worms and email viruses ;
- dissemination of information ;
- cascading failures.

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- $\mathbb{P}(X_i : 0 \to 1) = \beta \sum_{(i,j) \in E} X_j$, constant $\beta \in (0, 1]$;
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$$\rho(A) < \frac{1}{\beta} \quad \Rightarrow \quad \mathbb{E}[\tau] = O(\log n),$$

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Condition 2: Last Infection

$$\eta(m) > \frac{1}{\beta} \quad \Rightarrow \quad \log \mathbb{E}[\tau] = \Omega(n^{\alpha}) \text{ for some } \alpha > 0,$$

where $\eta(m)$ denotes generalized isoperimetric constant of G as

$$\eta(m) = \inf_{S \subset V, |S| \leq m} \frac{E(S, S^C)}{|S|}, \quad 0 < m \leq \left[\frac{n}{2}\right].$$

Supported Mathematical Theorems

Theorem 1

Suppose $\rho(A) < \frac{1}{\beta}$. Then, the probability that the epidemic has not died out by time t, given the initial condition $\mathbb{X}(0) \in \{0,1\}^V$, admits the following upper bound:

$$\mathbb{P}(\mathbb{X}(t) \neq 0) \leq \sqrt{n \parallel \mathbb{X}(0) \parallel_1} e^{(\beta\rho(A)-1)t},$$

where $\parallel \mathbb{X}(0) \parallel_1 = \sum_{i=1}^{n} X_i(0)$. In addition, under the condition 1, the time to extinction τ verifies

$$\mathbb{E}(\tau) \leq \frac{\log(n)+1}{1-\beta\rho(A)},$$

for any initial condition $\mathbb{X}(0)$.

Supported Mathematical Theorems

Theorem 2

Assume that the following inequality holds:

$$r := \frac{1}{\beta\eta(m)} < 1,$$

Then for any initial condition $\mathbb{X}(0)$ with $\sum_{i=1}^{n} X_i(0) > 0$, it holds that,

$$\mathbb{P}\left(\tau > \frac{[r^{-m+1}]}{2m}\right) \leq \frac{1-r}{e}(1 + O(r^m)).$$

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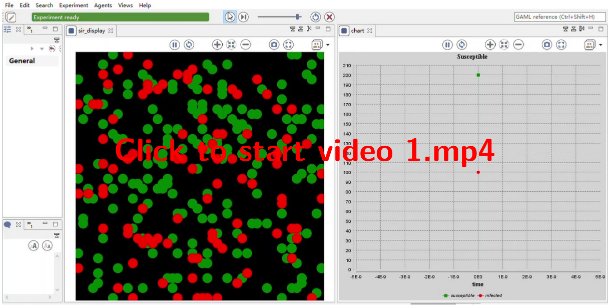
Experimental Result

- Ratio: $\frac{\text{infection}}{\text{cure}} = 1$;
- Initial condition: $\begin{cases} \text{No. of infected nodes} = 100 \\ \text{No. of infected nodes} = 200 \end{cases}$



Experimental Result

- Ratio: $\frac{\text{infection}}{\text{cure}} = 2.5$;
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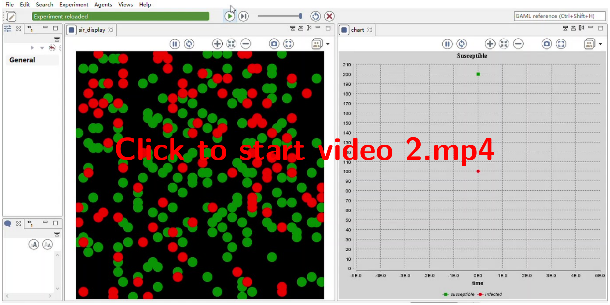

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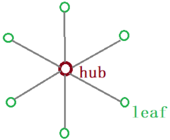
Star graph



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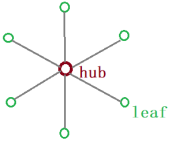
Star graph



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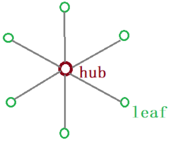

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2. $$\beta = n^{\alpha - \frac{1}{2}} \text{ for some } \alpha \in (0, \frac{1}{2}) \quad \Rightarrow \quad \log \mathbb{E}[\tau] = \Omega(n^{\alpha}).$$

Power law graph

In Power law graph, so-called scale-free graph, the probability $\mathbb{P}(k)$ that a vertex with k degree follows power law as

$$\mathbb{P}(k) \sim k^{-\gamma}, \quad \gamma > 1.$$

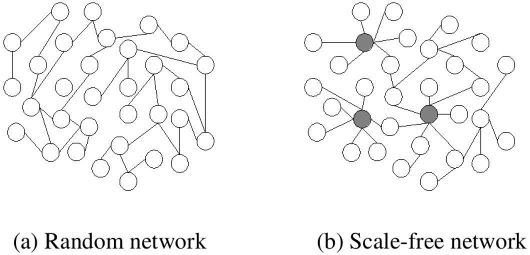

Figure: In the scale-free network, the larger hubs are highlighted[1]

[1] https://en.wikipedia.org/wiki/Scale-free_network

Power law graph

Topology conditions in power law graph can be obtained as

Theorem 4

Let m denote the maximum degree in the power law graph. For $\gamma \geq 2.5$,

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Let m denote the maximum degree and d denote average degree.
For $2 < \gamma < 2.5$,

1. for some $u \in (0, 1)$,

$$d\beta \frac{(\gamma-2)^2}{(\gamma-1)(3-\gamma)} \left(\frac{(\gamma-1)m}{(\gamma-2)d}\right)^{3-\gamma} < 1-u \quad \Rightarrow \quad \mathbb{E}[\tau] = O(\log n);$$

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2. let $\eta = d\frac{(\gamma-2)^2}{(\gamma-1)(3-1\gamma)}\left(\frac{(\gamma-1)m}{(\gamma-2)d}\right)^{3-\gamma}$. For $0 < \lambda < \frac{1}{\gamma-1}, 0 < u < 1$,

$$\begin{cases} \beta\eta > 1+u \\ \eta \gg \log n \left(\frac{d(\gamma-2)}{m(\gamma-1)}\right)^{\gamma-1} \end{cases} \quad \Rightarrow \quad \log \mathbb{E}[\tau] = \Omega(n^{1-\lambda(\gamma-1)}).$$

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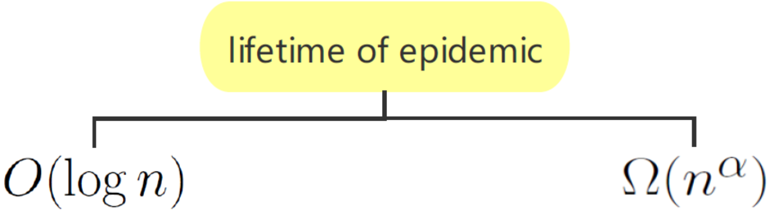
Summary



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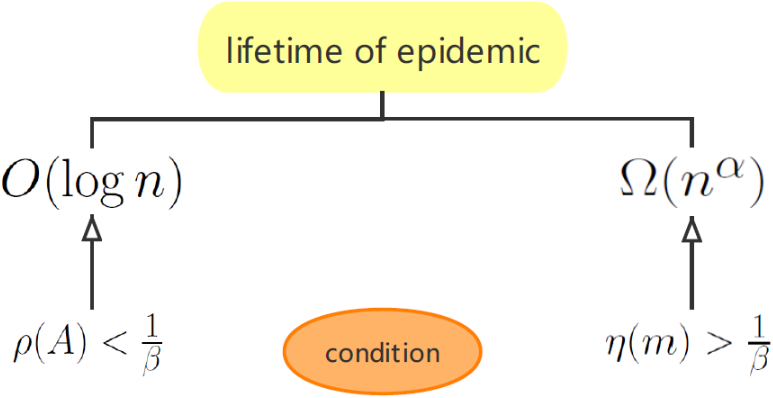
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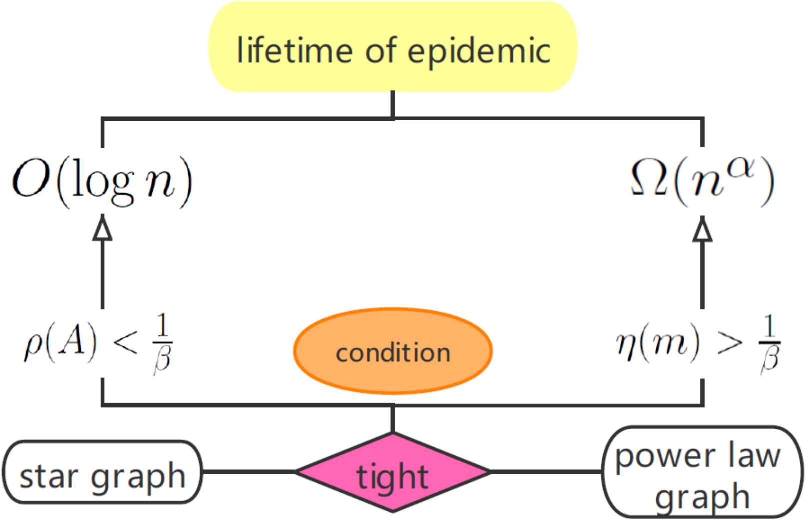
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Future work

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Thank you for your attention!

Questions?